Name of college - S.S. College, J-Bad
Deptt - Mathematics
Topic - Problem based on Raabe's Test contd (Infinite Series)
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Problem :→ Test for the convergence the series
$x^2(\log 2)^p + x^3(\log 3)^p + x^4(\log 4)^p + \cdots$

Solution → Given infinite series is
$x^2(\log 2)^p + x^3(\log 3)^p + x^4(\log 4)^p + \cdots$

$\therefore u_n = n$th Term of the series $= x^n(\log n)^p$

$u_{n+1} = (n+1)$th Term of the series

$$= x^{n+1}[\log(n+1)]^p$$

$$= x^{n+1}\left[\log n\left(1+\frac{1}{n}\right)\right]^p$$

$$= x^{n+1}\left[\log n + \log\left(1+\frac{1}{n}\right)\right]^p$$

$$= x^{n+1}\left[\log n + \frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n^2} + \frac{1}{3}\cdot\frac{1}{n^3}\cdots\right]^p$$

Now
$$\frac{u_n}{u_{n+1}} = \frac{(\log n)^p}{\left[\log n + \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\cdots\right]^p}\cdot\frac{x^n}{x^{n+1}}$$

$$= \left[\frac{\log n}{\log n\left\{1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \frac{1}{3n^3\log n} \cdots\right\}}\right]^p\cdot\frac{1}{x}$$

$$= \frac{1}{\left\{1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \frac{1}{3n^3\log n}\cdots\right\}^p}\cdot\frac{1}{x}$$

$$\frac{u_n}{u_{n+1}} = \left[1 + \left\{\frac{1}{n\log n} - \frac{1}{2n^2\log n} + \frac{1}{3n^3\log n} - \cdots\right\}\right]^{-p} \cdot \frac{1}{x}$$

$$\left[\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots\right]$$

$$= \left[1 - p\left\{\frac{1}{n\log n} - \frac{1}{2n^2\log n} - \cdots\right\} + \frac{(-p)(-p-1)}{\lfloor 2}\left\{\frac{1}{n\log n} + \cdots\right\} + \cdots\right]$$

$$(1+x)^{-n} = 1 - nx - \frac{n(-n-1)}{\lfloor 2} x^2 - \cdots$$

or $$\frac{u_n}{u_{n+1}} = \left[1 - \frac{p}{n\log n} + \frac{1}{2n^2}\left\{\frac{p}{\log n} + \frac{p(p+1)}{(\log n)^2} - \cdots\right\} + \cdots\right] \cdot \frac{1}{x}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

By ratio Test:-

If $\frac{1}{x} > 1 \Rightarrow$ the series is convergent.

or $x < 1 \Rightarrow$ the series is convergent.

Also $\frac{1}{x} < 1 \Rightarrow$ the series is Divergent.

$\Rightarrow x > 1 \rightarrow$ the series is Divergent.

If $\frac{1}{x} = 1$ i.e $x = 1$ Ratio Test Fails

If $x = 1$

then we have

$$\frac{u_n}{u_{n+1}} = 1 - \frac{p}{n\log n} + \frac{1}{2n^2}\left\{\frac{p}{\log n} + \frac{p(p+1)}{(\log n)^2} + \cdots\right]$$

$$n\left[\frac{u_n}{u_{n+1}}-1\right] = \frac{-p}{\log n} + \frac{1}{2n}\left\{\frac{p}{\log n} + \frac{p(p+1)}{(\log n)^2}\right\} + \cdots$$

$$\therefore \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty}\left[\frac{-p}{\log n} + \frac{1}{2n}\left\{\frac{p}{\log n} + \frac{p(p+1)}{(\log n)^2}\right\} + \cdots\right]$$

$$= 0 < 1$$

$\therefore$ From Raabe's test, the given series is Divergent.

$\therefore$ We conclude that the given series is convergent if $x<1$ and divergent if $x \geq 1$

<u>Problem 2</u>

Test for the convergence the series.

$$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \frac{1}{(\log 4)^p} + \cdots$$

<u>Solution</u> $\rightarrow$ General term of the series

$$u_n = \left\{\frac{1}{\log(n+1)}\right\}^p$$

$$u_{n+1} = \left\{\frac{1}{\log(n+2)}\right\}^p$$

$$\therefore \frac{u_{n+1}}{u_n} = \left\{\frac{\log(n+2)}{\log(n+1)}\right\}^p$$

$$= \left\{\frac{\log\{n(1+2/n)\}}{\log\{n(1+1/n)\}}\right\}^p = \left[\frac{\log n + \log(1+2/n)}{\log n + \log(1+1/n)}\right]^p$$

$$= \left[\frac{\log n + \log(1+2/n)}{\log n + \log(1+1/n)}\right]^p$$

$$= \left[\frac{\log n + 2/n - \frac{1}{2}\frac{2^2}{n^2} + \frac{1}{3}\frac{2^3}{n^3} - \cdots}{\log n + \frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n^2} + \frac{1}{3}\cdot\frac{1}{n^3}}\right]^p$$

$$= \left[\frac{\log n\left(1 + \frac{2}{n\log n} - \frac{1}{2}\cdot\frac{2^2}{n^2(\log n)} + \cdots\right)}{\log n\left[1 + \frac{1}{n\log n} - \frac{1}{2\log n}\frac{1}{n^2} + \cdots\right)}\right]^p$$

$$= \frac{\left[1 + \frac{2}{n\log n} - \right]^p}{\left[1 + \frac{1}{n\log n} - \right]^p}$$

$$= \left[1 + \frac{2}{n\log n} - \cdot\right]^p \left[1 + \frac{1}{n\log n} - \right]^{-p}$$

$$= \left[n + \frac{2p}{n\log n} + \cdots\right] \times \left[1 - \frac{p}{n\log n} - \cdots\right]$$

Expanding By Binomial theorem.

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left[1 + \frac{2p}{n\log n} + \cdots\right] \times \left[1 - \frac{p}{n\log n} + \cdots\right]$

$= 1$ Here $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$

$\Rightarrow$ Ratio Test Fails

$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{p}{n\log n} + \cdots\right]$

$= \frac{p}{\log n} + \cdots$

$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\left(\frac{p}{\log n} + \cdots\right) = 0 < 1$

thus By Raabe's Test, the given series $\sum u_n$ is divergent.

<u>Problem</u>

<u>Test For the Convergence of the Series</u>

$x\log x + x^2(\log 2x) + x^3(\log 3x) + \cdots + x^n(\log nx) + \cdots$

Solution $\rightarrow$ If $\sum u_n$ be the given Series then

$u_n = x^n(\log nx)$

and hence $u_{n+1} = x^{n+1}\log(n+1)x$

$= x^{n+1}\log(nx + x)$

$$\frac{u_n}{u_{n+1}} = \frac{x^n \log nx}{x^{n+1} \log\{(n+1)x\}}$$

$$= \left[\frac{\log nx}{\log(nx+x)}\right] \cdot \frac{1}{x}$$

$$= \left[\frac{\log(nx+x)}{\log nx}\right]^{-1} \cdot \frac{1}{x}$$

$$= \left[\frac{\log nx(1+\frac{1}{n})}{\log nx}\right]^{-1} \cdot \frac{1}{x}$$

$$= \left[\frac{\log nx + \log(1+1/n)}{\log nx}\right]^{-1} \frac{1}{x}$$

$$= \left[1 + \frac{\log(1+1/n)}{\log nx}\right]^{-1} \cdot \frac{1}{x}$$

use
$\log(1+x)$
$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \cdots$

$$= \left[1 + \frac{\frac{1}{n} - \frac{1}{2n^2} + \cdots}{\log nx}\right]^{-1} \frac{1}{x}$$

$$= \left[1 + \frac{1}{n \log nx} - \frac{1}{2n^2 \log nx} + \cdots\right] \frac{1}{x}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left[1 + \frac{1}{n \log nx} - \frac{1}{2n^2 \log nx} + \cdots\right] \cdot \frac{1}{x}$$

$$= \frac{1}{x}$$

From Ratio Test.

if $\frac{1}{x} > 1$ i.e $x < 1$ the Given Series is convergent.

if $\frac{1}{x} < 1$ i.e $x > 1$ the series is divergent.

if $\frac{1}{x} = 1$ i.e $x = 1$ this Ratio test fails

if $x = 1$

then $\frac{u_n}{u_{n+1}} = 1 - \frac{1}{n \log n} + 0 \cdot \frac{1}{n^2}$

$$\therefore n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[-\frac{1}{n \log n} + 0 \cdot \frac{1}{n^2}\right]$$

$$= -\frac{1}{\log n} + 0 \cdot \left(\frac{1}{n}\right)$$

$$\therefore \lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n \to 0}\left(-\frac{1}{\log n} + 0 \cdot \frac{1}{n}\right)$$

$$= 0 < 1$$

thus the series is divergent.

From above we conclude that the series is convergent if $x < 1$ and the series is divergent if $x \geq 1$